SO(n)-INVARIANT SPECIAL LAGRANGIAN SUBMANIFOLDS OF $\mathbb{C}^{n+1}$ WITH FIXED LOCI

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ABSTRACT. Let SO(n) act in the standard way on $\mathbb{C}^n$ and extend this action in the usual way to $\mathbb{C}^{n+1} = \mathbb{C} \oplus \mathbb{C}^n$.

It is shown that a nonsingular special Lagrangian submanifold $L \subset \mathbb{C}^{n+1}$ that is invariant under this SO(n)-action intersects the fixed $\mathbb{C} \subset \mathbb{C}^{n+1}$ in a nonsingular real-analytic arc A (which may be empty). If $n > 2$, then A has no compact component.

Conversely, an embedded, noncompact nonsingular real-analytic arc $A \subset \mathbb{C}$ lies in an embedded nonsingular special Lagrangian submanifold that is SO(n)-invariant. The same existence result holds for compact A if $n = 2$. If A is connected, there exist n distinct nonsingular SO(n)-invariant special Lagrangian extensions of A such that any embedded nonsingular SO(n)-invariant special Lagrangian extension of A agrees with one of these n extensions in some open neighborhood of A.

The method employed is an analysis of a singular nonlinear PDE and ultimately calls on the work of Gérard and Tahara to prove the existence of the extension.

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1. Introduction

1.1. **Special Lagrangian geometry.** In the now-classic 1982 paper [3] of Harvey and Lawson, an m-dimensional submanifold $L \subset \mathbb{C}^m$ is defined to be *special Lagrangian* if it is Lagrangian with respect to the standard Kähler form

$$\omega = \frac{\mathrm{i}}{2}\left(\mathrm{d}z_1 \wedge \mathrm{d}\overline{z_1} + \cdots + \mathrm{d}z_m \wedge \mathrm{d}\overline{z_m}\right) \tag{1.1}$$

and, moreover, the standard holomorphic volume form $\mathrm{d}z = \mathrm{d}z_1 \wedge \cdots \mathrm{d}z_m$ pulls back to L to be a real n-form, i.e., $\Upsilon = \mathrm{Im}(\mathrm{d}z)$ pulls back to L to be identically zero.[1]

In other words, an m-dimensional submanifold $L \subset \mathbb{C}^m$ is special Lagrangian if and only if it is an integral manifold of the ideal $\mathcal{I}$ generated by the differential forms ω and Υ.

As Harvey and Lawson show, an $L \subset \mathbb{C}^m$ is special Lagrangian if and only if it is calibrated by the m-form $\Phi = \mathrm{Re}(\mathrm{d}z)$. In particular, special Lagrangian submanifolds are mass-minimizing.

In the intervening 20 years, special Lagrangian submanifolds and foliations whose leaves are special Lagrangian submanifolds (and their generalizations to Calabi-Yau manifolds in the place of $\mathbb{C}^m$) have turned out to be important in several areas in differential geometry and theoretical physics. The interested reader can consult recent papers of Joyce, such as [4] for a survey and references to the (by now rather large) literature on this subject.

1.2. **Symmetry reduction.** Because understanding the possible types of singularities of special Lagrangian submanifolds is important for applications, considerable effort has been expended to construct explicit examples.

One of the standard methods of constructing explicit integral manifolds of an ideal $\mathcal{I}$ is to use symmetry reduction, i.e., to fix a subgroup G of the symmetries of the given ideal $\mathcal{I}$ and look for solutions that are invariant under the action of G.

1.2.1. *Symmetries of $\mathcal{I}$.* In the case of special Lagrangian geometry, the symmetry group of the ideal $\mathcal{I}$ depends to some extent on the dimension m.

The dimension $m = 2$ is exceptional. In this case, there exists a complex structure J on $\mathbb{C}^2$ (different from the standard one) such that ω and Υ are the real and imaginary parts, respectively, of a J-holomorphic $(2,0)$-form on $\mathbb{C}$. Thus, the local symmetries of $\mathcal{I}$ are simply the local biholomorphisms of $(\mathbb{C}^2, J)$. (Moreover, the special Lagrangian submanifolds are simply the J-complex curves in $(\mathbb{C}^2, J)$.)

For $m > 2$, the symmetry group of the ideal $\mathcal{I}$ is finite dimensional and consists of the group generated by the translations in $\mathbb{C}^m$, the group $\mathrm{SU}(m)$ acting linearly on $\mathbb{C}^m$, the dilations by nonzero complex numbers λ that satisfy $\lambda^m \in \mathbb{R}$, and conjugation.

[1] In fact, Harvey and Lawson show that if $L \subset \mathbb{C}^m$ is Lagrangian and oriented, then the pullback of $\mathrm{d}z$ to L is of the form $\mathrm{e}^{\mathrm{i}\phi}\,\mathrm{vol}_L$, for some function $\phi : L \to S^1$, where vol_L is the n-form on L that is the volume form of the induced metric on L. The 'phase factor' $\mathrm{e}^{\mathrm{i}\phi}$ is a sort of complex determinant, so the equation $\mathrm{e}^{\mathrm{i}\phi} = 1$ can be thought of as setting a determinant equal to 1, hence the modifier 'special'.

1.2.2. *Cohomogeneity one examples.* Harvey and Lawson themselves considered and solved the problem of describing the G-invariant special Lagrangian submanifolds of $\mathbb{C}^m$ when $G \subset \mathrm{SU}(m)$ is either $\mathrm{SO}(m)$ or $\mathbb{T}^{m-1}$, a maximal torus in $\mathrm{SU}(m)$. The Lagrangian-isotropic orbits of these actions have dimension $m-1$, and so the standard theory leads one to expect that, in these cases, the G-invariant special Lagrangian manifolds will be found by solving a single ODE.

Indeed, this is precisely what Harvey and Lawson find. They show that, when $G = \mathrm{SO}(m)$, the sets

$$L_c = \{\zeta\,\mathbf{u} \mid \mathbf{u} \in S^{m-1},\ \zeta \in \mathbb{C},\ \mathrm{Im}(\zeta^n) = c\} \tag{1.2}$$

for c a real constant are $\mathrm{SO}(m)$-invariant and special Lagrangian at smooth points. The set L_0 is the union of m special Lagrangian m-planes that are each individually $\mathrm{SO}(m)$-invariant. When $c \neq 0$, the set L_c, which is smooth, is the disjoint union of m connected components, each of which is diffeomorphic to $\mathbb{R} \times S^{m-1}$ with each 'end' asymptotic to a $SO(m)$-invariant special Lagrangian m-plane. Moreover, any smooth, connected $\mathrm{SO}(m)$-invariant special Lagrangian submanifold of $\mathbb{C}^m$ is an open subset of one of these examples.

In the case of $G = \mathbb{T}^{m-1}$, the picture is slightly more complicated: The $\mathbb{T}^{m-1}$-invariant special Lagrangian submanifolds are the simultaneous level sets of the functions $f_0, \ldots, f_{m-1}$ where

$$\begin{aligned} f_0 &= \mathrm{Im}\big(z_1 z_2 \cdots z_m\big) \\ f_k &= |z_k|^2 - |z_m|^2, \qquad 1 \le k < m. \end{aligned} \tag{1.3}$$

Many of these level sets are singular (some are analytically irreducible and some are not) and they furnish interesting examples of the kinds of singularities that mass minimizing currents can display. (In fact, this is one of the reasons that Harvey and Lawson found them so interesting.)

1.2.3. *General compact group actions.* It may clarify matters to consider special Lagrangian symmetry reduction in the case of an arbitrary connected compact subgroup G, that, without loss of generality, can be assumed to be a subgroup of $\mathrm{SU}(m)$.

Recall that the Lie algebra $\mathfrak{su}(m)$ of $\mathrm{SU}(m)$ is the vector space of m-by-m skew-Hermitian matrices and is endowed with a positive definite $\mathrm{Ad}\big(\mathrm{SU}(m)\big)$-invariant inner product defined by $\langle a, b\rangle = -\operatorname{tr}\big(ab\big)$.

Let $\mathfrak{g} \subset \mathfrak{su}(m)$ denote the Lie algebra of G and write $\mathfrak{g} = \mathfrak{z} \oplus [\mathfrak{g}, \mathfrak{g}]$ where $\mathfrak{z} \subset \mathfrak{g}$ is the center of $\mathfrak{g}$. Using the above inner product, $\mathfrak{g}$ can be identified with its dual, so $\mathfrak{g}^*$ and $\mathfrak{g}$ will be identified henceforth. Let $\pi_{\mathfrak{g}} : \mathfrak{su}(m) \to \mathfrak{g}$ denote the orthogonal projection onto $\mathfrak{g}$.

The linear action of G on $\mathbb{C}^m$ is ω-Poisson, with momentum mapping $\mu : \mathbb{C}^m \to \mathfrak{g} = \mathfrak{g}^*$ given by

$$\mu(z) = \pi_{\mathfrak{g}}\big(\mathrm{i}\, z\, {}^t\bar{z}\,\big). \tag{1.4}$$

Note that a G-orbit $G \cdot z \subset \mathbb{C}^m$ is ω-isotropic if and only if $\mu(z)$ lies in $\mathfrak{z}$.

1.2.4. *Smooth reduction.* For a fixed $\xi \in \mathfrak{z}$, let $\mu^{-1}(\xi)^* \subset \mu^{-1}(\xi) \subset \mathbb{C}^m$ denote the subset that consists of μ-clean points.[2]

[2]For a smooth mapping $f : X \to Y$ between smooth manifolds, a point $x \in X$ is said to be f-*clean* if the level set $Z = f^{-1}\big(f(x)\big) \subset X$ is a smooth submanifold near x and $T_x Z = \ker f'(x)$.

The submanifold $\mu^{-1}(\xi)^*$ is usually dense in $\mu^{-1}(\xi)$ and has dimension $m+k$ for some $k \geq 1$. The G-orbits in $\mu^{-1}(\xi)^*$ are ω-isotropic and of dimension $m-k$.

The symplectic quotient $M^*_\xi = G\backslash\mu^{-1}(\xi)^*$ will, at most points, be a smooth symplectic manifold of dimension $2k$, with the canonical projection $\pi_\xi : \mu^{-1}(\xi)^* \to M^*_\xi$ being a smooth submersion and the induced symplectic form ω_ξ on M^*_ξ having the property that $\pi^*_\xi(\omega_\xi)$ is equal to the pullback of ω to $\mu^{-1}(\xi)^*$.

The reduced space M^*_ξ inherits a metric g_ξ defined by the condition that $\pi_\xi : \mu^{-1}(\xi)^* \to M^*_\xi$ be a Riemannian submersion. The pair (g_ξ, ω_ξ) then define a Kähler structure on M^*_ξ.

For any ω_ξ-Lagrangian submanifold $L_\xi \subset M_\xi$, its preimage ${\pi_\xi}^{-1}(L_\xi) \subset \mu^{-1}(\xi)^* \subset \mathbb{C}^m$ is a G-invariant ω-Lagrangian submanifold of $\mathbb{C}^m$.

Moreover, it is not difficult to show that there is a k-form Υ_ξ defined on M^*_ξ (only up to a sign if $\mu^{-1}(\xi)^*$ is not orientable) with the property that Υ_ξ vanishes when pulled back to L_ξ if and only if ${\pi_\xi}^{-1}(L_\xi)$ is special Lagrangian. In fact, up to an orientation, Υ_ξ can be seen as the imaginary part of a $(k,0)$-form on the Kähler manifold M^*_ξ.

Thus, away from the singularities of the various mappings, the problem of describing the G-invariant special Lagrangian submanifolds of $\mathbb{C}^m$ can be reduced to a similar problem in lower dimensions.

Example 1 (Cohomogeneity 1). For example, when $k = 1$ (as is true in the Harvey-Lawson examples), M^*_ξ is a surface and Υ_ξ is a 1-form on M^*_ξ. The problem of describing the G-invariant special Lagrangian submanifolds (away from singularities) is thus reduced to finding the integral curves of a 1-form on a surface.

As another example, to illustrate the method and because this will be used to construct a needed example later, consider the action of $G = S^1 \times \mathrm{SO}(p) \times \mathrm{SO}(q)$ on $\mathbb{C}^{p+q}$ acting on $\mathbb{C}^{p+q}$ via the action

$$(e^{\mathrm{i}\theta}, A, B) \cdot (z, w) = \left(e^{q\mathrm{i}\theta} Az,\, e^{-p\mathrm{i}\theta} Bw\right), \qquad z \in \mathbb{C}^p,\ w \in \mathbb{C}^q. \tag{1.5}$$

The momentum mapping is

$$\mu(z, w) = \left(\mathrm{i}(q|z|^2 - p|w|^2), \mathrm{Im}(z\,{}^t\overline{z}), \mathrm{Im}(w\,{}^t\overline{w})\right). \tag{1.6}$$

Taking $\xi = (\mathrm{i}c, 0, 0)$ for some constant $c \in \mathbb{R}$, (which is the only allowable choice unless p or q is 2) yields

$$\mu^{-1}(\xi) = \left\{\, (\,\zeta\,\mathbf{u},\, \eta\,\mathbf{v}\,) \,\middle|\, \mathbf{u} \in S^{p-1},\, \mathbf{v} \in S^{q-1},\, \zeta, \eta \in \mathbb{C},\, q|\zeta|^2 - p|\eta|^2 = c \,\right\}. \tag{1.7}$$

One then finds that $\Upsilon_\xi = \mathrm{d}\big(\mathrm{Re}(\zeta^q\eta^p)\big)$, so that $\mu^{-1}(\xi)$ is 'foliated' (some of the 'leaves' may be singular) by the level sets of the function $\mathrm{Re}(\zeta^q\eta^p)$, which are special Lagrangian.

Example 2 (Cohomogeneity 2 and almost complex surfaces). The $k = 2$ case is somewhat more interesting. In this case, assuming that $\mu^{-1}(\xi)^*$ is orientable,[3] it is not difficult to show that M^*_ξ (which has real dimension $2k = 4$), inherits a natural almost complex structure J_ξ such that $L_\xi \subset M^*_\xi$ is a J_ξ-complex curve if and only if ${\pi_\xi}^{-1}(L_\xi)$ is special Lagrangian. (In fact, ω_ξ and Υ_ξ in this case turn out to be essentially the real and imaginary parts of a $(2,0)$-form with respect to the almost complex structure J_ξ.) If $\mu^{-1}(\xi)^*$ is not orientable, then by passing to

[3]The nonorientable case does occur, as will be seen below.

its orientation double cover, one can define Υ_ξ and J_ξ as before, but on a covering space of M^*_ξ.

1.2.5. *Singularities in reduction.* Thus, one understands, in a general way, how to describe the special Lagrangian submanifolds invariant under a compact Lie group, at least away from singular points of the quotient $M_\xi = G\backslash\mu^{-1}(\xi)$. However, in the case of cohomogeneity greater than 1, as in the recent work of Dominic Joyce [4] (cf., especially, the series of papers [5, 6, 7]), the singular locus plays an important role. The standard approach described above is not adequate to address existence and uniqueness questions in this situation.

To take a specific example, Joyce considers the case of $G = \mathrm{SO}(2)$ acting on $\mathbb{C}^3 = \mathbb{C}^1 \oplus \mathbb{C}^2$ (trivially on the first summand $\mathbb{C}^1$ and in the standard way on the second summand $\mathbb{C}^2$).[4]

Joyce showed that a nonsingular SO(2)-invariant special Lagrangian 3-fold $L \subset \mathbb{C}^3$ meets the fixed factor $\mathbb{C}^1$, if at all, in a nonsingular real-analytic arc $A \subset \mathbb{C}^1$.

Now, in fact, A lies in two distinct SO(2)-invariant, special Lagrangian 3-folds, namely L and $\mathrm{i}{\star}L$, where i acts on $\mathbb{C}^3$ as $\mathrm{i}{\star}(z_0, z_1, z_2) = (\,z_0,\, \mathrm{i}\,z_1,\, \mathrm{i}\,z_2\,)$. (That $\mathrm{i}{\star}L$ is special Lagrangian and distinct from L is left to the reader, but see §2.3.1.)

For use in his work on singularities of special Lagrangian 3-folds, Joyce asked[5] whether, any nonsingular SO(2)-invariant special Lagrangian 3-fold that meets $\mathbb{C}$ in the arc $A \subset \mathbb{C}^1$ is equal to one of L or $\mathrm{i}{\star}L$ in some neighborhood of A.

1.3. **Results.** As will be proved in this article, a stronger statement is true: For any nonsingular, connected, embedded, real-analytic arc $A \subset \mathbb{C}$, there exist two distinct SO(2)-invariant, nonsingular, embedded, connected special Lagrangian submanifolds of $\mathbb{C}^3$ that intersect the fixed $\mathbb{C}^1$ in the analytic arc A. Moreover, any SO(2)-invariant, nonsingular, embedded, connected special Lagrangian submanifold of $\mathbb{C}^3$ that intersects the fixed $\mathbb{C}^1$ in the analytic arc A agrees with one of these two special Lagrangian submanifolds in some open neighborhood of A.

Moreover, these existence and uniqueness results generalize when appropriately stated, when A is noncompact,[6] to the case of $\mathrm{SO}(n)$ acting on $\mathbb{C}^{n+1} = \mathbb{C}^1 \oplus \mathbb{C}^n$ (trivially on the first summand $\mathbb{C}^1$ and in the standard way on the second summand $\mathbb{C}^n$). As this is no more difficult than the case $n = 2$ (which is the one of interest to Joyce), these more general results will be proved in this article, cf. Theorems 2 and 3.

The method used to establish this uniqueness result is basically an elementary, though slightly delicate, examination of a power series expansion. However, the this argument, while it proves uniqueness, only suffices to prove existence in the category of formal power series. It is not adequate to address the problem of existence and the elementary argument does not seem to lend itself to any of the usual methods of proving convergence of the formal power series. However, it turns out that, by calling on the work of Gérard and Tahara on existence of solutions of certain kinds of singular holomorphic PDE, the convergence of this power series can be established.

[4]The SO(2)-action that Joyce actually considers is conjugate to this one in SU(3), but this obviously will not affect the results.

[5]private communication, 2 January 2002.

[6]When $n > 2$, the assumption of noncompactness is necessary, cf. Proposition 1.

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After Version 1 of this article was posted to the arXiv on 12 February 2004, Professors Castro and Urbano made me aware of their preprint [1], in which non-singular reduction in the case of $\mathrm{SO}(p)\times\mathrm{SO}(q)$ acting on $\mathbb{C}^{p+q}$ is discussed at greater length. I thank them for bringing their article to my attention.

2. Invariant special Lagrangians

Write $z_j = x_j + \mathrm{i}\, y_j$ $(0 \le j \le n)$ for the real and imaginary parts of the complex coordinates in $\mathbb{C}^{n+1}$. The Kähler form is

$$\omega = \tfrac{\mathrm{i}}{2}\left(\mathrm{d}z_0\wedge\mathrm{d}\overline{z_0} + \mathrm{d}z_1\wedge\mathrm{d}\overline{z_1} + \cdots + \mathrm{d}z_n\wedge\mathrm{d}\overline{z_n}\right) \tag{2.1}$$

and the imaginary part of the complex volume form is

$$\Upsilon = \mathrm{Im}\big(\mathrm{d}z_0\wedge\mathrm{d}z_1\wedge\cdots\mathrm{d}z_m\big) \tag{2.2}$$

By definition, a (real) $(n{+}1)$-dimensional submanifold $L\subset\mathbb{C}^{n+1}$ is special Lagrangian if and only if both ω and Υ vanish when pulled back to L.

2.1. The group action.

The group $\mathrm{SO}(n)$ will be taken to act on $\mathbb{C}^{n+1}=\mathbb{C}\oplus\mathbb{C}^n$ in the manner as described in the introduction, namely, trivially on the first $\mathbb{C}$-summand and as the $\mathbb{C}$-linear extension of its standard action on $\mathbb{R}^n$ in the remaining $\mathbb{C}^n$.

2.2. The reductions.

The momentum mapping of this action $\mu:\mathbb{C}^{n+1}\to\mathfrak{so}(n)$ is then (up to a constant scale factor that will be irrelevant in what follows)

$$\mu(z_0,\ldots,z_n) = (x_i y_j - y_i x_j)_{1\le i,j\le n}\in\mathfrak{so}(n). \tag{2.3}$$

2.2.1. *Nonzero momentum in the case $n=2$.* When $n>2$, the algebra $\mathfrak{so}(n)$ has trivial center, so, by the remarks in §1.2.3, there is only need to consider $\mu^{-1}(0)$, which is the case of most concern in this article.

However, when $n=2$, the algebra $\mathfrak{so}(2)\simeq\mathbb{R}$ is abelian and the level sets of μ are the hypersurfaces H_c of the form $x_1y_2-y_1x_2=c$, where $c\in\mathbb{R}$ is any constant.

When $c\neq0$, the μ-level set $H_c\subset\mathbb{C}^3$ is smooth and $\mathrm{SO}(2)\simeq S^1$ acts freely on M_c. The reduced space $M_c=\mathrm{SO}(2)\backslash H_c$ is thus a smooth 4-manifold endowed with a nonintegrable almost complex structure J_c such that the J_c-complex curves in M_c are the $\mathrm{SO}(2)$-quotients of the $\mathrm{SO}(2)$-invariant special Lagrangian 3-folds in $\mathbb{C}^3$ that lie in the level set H_c.

Each of the (M_c,J_c) with $c\neq0$ is equivalent to (M_1,J_1), so there is really only one case of nonzero momentum that needs to be treated. In any case, since there are no singular $\mathrm{SO}(2)$-orbits involved, these cases are reduced to the study of complex curves in almost complex 4-manifolds and will not be discussed further here.

2.2.2. *The zero momentum almost complex structure.* Henceforth, to avoid having to continually mention trivial cases, it will be assumed that $n > 1$.

The locus $\mu^{-1}(0) \subset \mathbb{C}^{n+1}$ is singular, with its singular locus consisting of the fixed points of the $\mathrm{SO}(n)$-action, i.e., the points of the form $(z_0, 0, \ldots, 0)$. It can be parametrized via the mapping $\Phi : \mathbb{C} \times \mathbb{C} \times S^{n-1} \to \mu^{-1}(0)$ defined by

$$\Phi(w, \zeta, \mathbf{u}) = (w, \zeta\, u_1, \zeta\, u_2, \ldots, \zeta u_n) \tag{2.4}$$

where $w, \zeta \in \mathbb{C}$ and $\mathbf{u} = (u_1, \ldots, u_n) \in S^{n-1}$. Note that $\Phi(w, \zeta, \mathbf{u}) = \Phi(w, -\zeta, -\mathbf{u})$, so that Φ is a 2-to-1 diffeomorphism away from the locus $\zeta = 0$, which is mapped by Φ onto the fixed locus of the $\mathrm{SO}(n)$-action.

Computation yields

$$\Phi^*(\omega) = \tfrac{\mathrm{i}}{2}\bigl(\mathrm{d}w \wedge \mathrm{d}\bar{w} + \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}\bigr) \tag{2.5}$$

and

$$\Phi^*(\Upsilon) = \tfrac{1}{n}\mathrm{Im}\bigl(\mathrm{d}w \wedge \mathrm{d}(\zeta^n)\bigr) \wedge \Omega \tag{2.6}$$

where Ω is the standard volume form on S^{n-1}, i.e.,

$$\Omega = u_1\, \mathrm{d}u_2 \wedge \ldots \wedge \mathrm{d}u_n + \cdots + (-1)^{n-1} u_n\, \mathrm{d}u_1 \wedge \ldots \wedge \mathrm{d}u_{n-1}\,. \tag{2.7}$$

It follows that any $\mathrm{SO}(n)$-invariant special Lagrangian $L \subset \mathbb{C}^{n+1}$ (of momentum zero in case $n = 2$) that does not meet the fixed locus is of the form $L = \Phi\bigl(\Sigma \times S^{n-1}\bigr)$ for some surface $\Sigma \subset \mathbb{C}^2$ that does not meet the line $\zeta = 0$ and that is an integral manifold of the two 2-forms

$$\begin{aligned} \Psi_1 &= \tfrac{\mathrm{i}}{2}\bigl(\mathrm{d}w \wedge \mathrm{d}\bar{w} + \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}\bigr), \\ \Psi_2 &= \mathrm{Im}\bigl(\mathrm{d}w \wedge \mathrm{d}(\zeta^n)\bigr). \end{aligned} \tag{2.8}$$

Let $M_0 \subset \mathbb{C}^2$ denote the complement of the line $\zeta = 0$. Then Ψ_1 and Ψ_2 are linearly independent on M_0.

Moreover, on M_0, the 2-forms Ψ_1 and Ψ_2 are multiples of the real and imaginary parts of the decomposable complex-valued 2-form

$$\Psi = \omega_1 \wedge \omega_2 = -\frac{2}{n|\zeta|^{n-1}}\, \Psi_2 + 2\mathrm{i}\, \Psi_1 \tag{2.9}$$

where ω_1 and ω_2 are the 1-forms on M_0 defined by the formulae

$$\omega_1 = \mathrm{d}w + \mathrm{i}\, \frac{\overline{\zeta}^{n-1}\, \mathrm{d}\overline{\zeta}}{|\zeta|^{n-1}}, \qquad \omega_2 = \mathrm{d}\overline{w} + \mathrm{i}\, \frac{\zeta^{n-1}\, \mathrm{d}\zeta}{|\zeta|^{n-1}}. \tag{2.10}$$

Since $\omega_1 \wedge \omega_2 \wedge \overline{\omega_1} \wedge \overline{\omega_2} \neq 0$, it follows that there exists a unique almost complex structure J_0 on M_0 for which ω_1 and ω_2 furnish a basis of $\Omega^{1,0}(M_0, J_0)$.

Remark 1 (Nonintegrability of J_0). Although it will not be needed in the rest of this article, the reader might like to know a bit more about the almost complex structure J_0, so some information about it will be mentioned here.

First, as is easily computed, J_0 is not integrable. In fact, its Nijenhuis tensor is nowhere vanishing.

Second, the almost complex manifold (M_0, J_0) is homogeneous: The diffeomorphisms $F_{a,b} : M_0 \to M_0$ defined by

$$F_{a,b}(w, \zeta) = \left(\frac{\overline{b}^n}{|b|^{n-1}}\, w + a,\ b\, \zeta \right) \tag{2.11}$$

for constants $a \in \mathbb{C}$ and $b \in \mathbb{C}^*$ preserve the almost complex structure J_0 and it is clear that these mappings generate a group that acts simply transitively on M_0.

It is not difficult to show that the mappings $F_{a,b}$ as defined in (2.11) together with the involution $A : M_0 \to M_0$ defined by

$$A(w, \zeta) = (\,\overline{w}, \,\overline{\zeta}\,) \tag{2.12}$$

generate the group of automorphisms of (M_0, J_0). In fact, a slightly stronger statement is true: Any local automorphism of (M_0, J_0) defined on a connected open subset of M_0 extends uniquely to a global automorphism and is either of the form $F_{a,b}$ or $A{\circ}F_{a,b}$.

This last claim follows from the properties of the $(1,0)$-forms

$$\eta_1 = \frac{|\zeta|^{n-1}}{\overline{\zeta}^n}\,\mathrm{d}w + \mathrm{i}\,\frac{\mathrm{d}\overline{\zeta}}{\overline{\zeta}}\,, \qquad \eta_2 = \frac{|\zeta|^{n-1}}{\zeta^n}\,\mathrm{d}\overline{w} + \mathrm{i}\,\frac{\mathrm{d}\zeta}{\zeta}\,, \tag{2.13}$$

which are invariant under the action of $F_{a,b}$ and are exchanged by A.

Inspection shows that the mapping $C : M_0 \to M_0$ defined by

$$C(w, \zeta) = (w, \mathrm{e}^{\mathrm{i}\pi/n}\zeta) \tag{2.14}$$

is J_0-antilinear (since $C^*\omega_1 = \overline{\omega_2}$ and $C^*\omega_2 = \overline{\omega_1}$), a fact that will be useful below.

Third, the almost complex structure J_0 on M_0 cannot be extended continuously across the line $\zeta = 0$. However, in view of the fact that this line is the fixed locus of the 'conjugation' C, one can think of the line $\zeta = 0$ as a sort of 'singular' totally real submanifold of $(\mathbb{C}^2, J_0)$.

Remark 2 (The double cover). The reader may have noticed that M_0 is not the symplectic quotient $\mathrm{SO}(n)\backslash\mu^{-1}(0)^*$ but rather is a double cover of it. In fact, when n is odd, it is necessary to take this double cover since $\mu^{-1}(0)^*$ is not orientable when n is odd. Consequently, in this case, J_0 is only defined up to a sign on $\mathrm{SO}(n)\backslash\mu^{-1}(0)^*$.

There are other reasons for working on M_0. As will be seen below, a nonsingular $\mathrm{SO}(n)$-invariant special Lagrangian submanifold $L \subset \mathbb{C}^{n+1}$ that meets the fixed locus will be represented by a smooth surface $\Sigma \subset M_0$ that extends across the line $\zeta = 0$ to a smoothly embedded surface in $\mathbb{C}^2$ that meets the line $\zeta = 0$ in a smooth analytic arc.

2.3. **Invariant special Lagrangian planes.** Suppose that a nonsingular $\mathrm{SO}(n)$-invariant special Lagrangian $L \subset \mathbb{C}^{n+1}$ meets the fixed locus $\mathbb{C}$ of the $\mathrm{SO}(n)$-action at a point $z \in L$. Then the tangent plane T_zL must be a $\mathrm{SO}(n)$-invariant special Lagrangian $(n{+}1)$-plane.

There is a circle of such $(n{+}1)$-planes: For each ψ, there is the $(n{+}1)$-plane P_ψ defined by the linearly independent equations

$$\begin{aligned} 0 &= \cos n\psi\,\mathrm{d}y_0 + \sin n\psi\,\mathrm{d}x_0 \\ &= \cos\psi\,\mathrm{d}y_1 - \sin\psi\,\mathrm{d}x_1 = \cdots = \cos\psi\,\mathrm{d}y_n - \sin\psi\,\mathrm{d}x_n\,. \end{aligned} \tag{2.15}$$

Of course $P_\psi = P_{\psi+\pi}$, but the planes $\{P_\psi \mid 0 \le \psi < \pi\}$ are pairwise distinct.

Note that each P_ψ intersects $\mathbb{C}$ in a real 1-dimensional linear subspace; that, conversely, each (real) 1-dimensional linear subspace of $\mathbb{C}$ lies in exactly n of these $\mathrm{SO}(n)$-invariant special Lagrangian $(n{+}1)$-planes; and that the projections of these n special Lagrangian $(n{+}1)$-planes into $\mathbb{C}^n$ are pairwise disjoint.

2.3.1. *The λ-action.* Let $\lambda = \mathrm{e}^{\pi \mathrm{i}/n}$, so that λ generates a multiplicative cyclic subgroup of order $2n$, denoted $\mathbb{Z}_{2n} \subset S^1 \subset \mathbb{C}$.

The $\mathbb{Z}_{2n}$-action on $\mathbb{C}^{n+1}$ defined by

$$\lambda^j{\star}(z_0, , z_1, \ldots, z_n) = \bigl(z_0,\, \lambda^j z_1,\, \ldots,\, \lambda^j z_n\bigr) \tag{2.16}$$

pulls back the holomorphic volume form $\mathrm{d}z$ to $(-1)^j \mathrm{d}z$ and commutes with the $\mathrm{SO}(n)$-action. It follows that this action carries special Lagrangian $(n{+}1)$-planes to other special Lagrangian $(n{+}1)$-planes (but may reverse their orientations) and permutes the n $\mathrm{SO}(n)$-invariant special Lagrangian $(n{+}1)$-planes that contain a given fixed (real) line in $\mathbb{C}$.

In particular, if $L \subset \mathbb{C}^{n+1}$ is special Lagrangian, then $\lambda^j{\star}L$ is also special Lagrangian for $0 \leq j < 2n$ and satisfies $\lambda^j{\star}L \cap \mathbb{C} = L \cap \mathbb{C}$. Note that $\lambda^{j+n}{\star}L$ and $\lambda^j{\star}L$ are tangent along their common intersection with $\mathbb{C}$. As will be seen below in Remark 3, these two special Lagrangian submanifolds are actually equal in a neighborhood of $\mathbb{C}$.

2.3.2. *A commuting action.* The $\mathrm{SO}(n)$-action commutes with the group of special Lagrangian symmetries of $\mathbb{C}^{n+1}$ of the form

$$\Phi_{a,\theta}(z_0, z_1, \ldots, z_n) = \bigl(\mathrm{e}^{n\mathrm{i}\theta}\, z_0 + a,\, \mathrm{e}^{-\mathrm{i}\theta}\, z_1,\, \ldots,\, \mathrm{e}^{-\mathrm{i}\theta}\, z_n, \bigr). \tag{2.17}$$

where $a \in \mathbb{C}$ and $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$ are constants. Note that this group action (which *does* preserve the special Lagrangian calibration) acts transitively on the $(n{+}1)$-planes of the form P_ψ. Also, this group action preserves the fixed locus $\mathbb{C}$ and acts on $\mathbb{C}$ as the group of Euclidean isometries of $\mathbb{C}$ (regarded as a real 2-plane).

2.4. **The fixed locus.** The goal of this subsection is to examine the geometry of an $\mathrm{SO}(n)$-invariant special Lagrangian submanifold $L \subset \mathbb{C}^n$ that meets the fixed locus $\mathbb{C} \subset \mathbb{C}^{n+1}$ then apply the results to prove the following (which was proved by Dominic Joyce in the case $n = 2$):

Proposition 1. *Suppose that $L \subset \mathbb{C}^{n+1}$ is an embedded nonsingular special Lagrangian submanifold that is $\mathrm{SO}(n)$-invariant and that meets the fixed locus $\mathbb{C}$. Then $A = L \cap \mathbb{C}$ is an embedded nonsingular real-analytic curve in $\mathbb{C}$. If $n > 2$, then A has no compact component.*

Proof. Let $L \subset \mathbb{C}^{n+1}$ be an embedded, nonsingular $\mathrm{SO}(n)$-invariant special Lagrangian submanifold that meets the fixed line $\mathbb{C} \subset \mathbb{C}^{n+1}$ at a point $z \in L \cap \mathbb{C}$.

After applying an action of the form (2.17), it can be assumed that $z = 0$ and that $T_0L = P_0 = \mathbb{R}^{n+1}$.

Since L is embedded and Lagrangian and since $T_0L = \mathbb{R}^{n+1}$, it follows that, in some neighborhood of $0 \in \mathbb{C}^{n+1}$, the submanifold L can be parametrized in the form

$$\left(x_0 + \mathrm{i}\,\frac{\partial F}{\partial x_0}(x_0, x_1, \ldots, x_n),\, \ldots,\, x_n + \mathrm{i}\,\frac{\partial F}{\partial x_n}(x_0, x_1, \ldots, x_n),\right) \tag{2.18}$$

for some function F that is defined on a neighborhood of $0 \in \mathbb{R}^{n+1}$ and has all of its first and second partials vanishing there. The function F can be made unique by requiring that $F(0, \ldots, 0) = 0$, so assume this.

Since L is nonsingular and special Lagrangian (and hence minimal), the known regularity of minimal submanifolds [8] implies that L is real-analytic and hence that F is real-analytic also.

Because F is invariant under the action of $\mathrm{SO}(n)$, there exists a real-analytic function ϕ defined in a neighborhood $V \subset \mathbb{R}^2$ of $(0,0)$ that is even in the second variable (i.e., $\phi(t,\sigma) = \phi(t,-\sigma)$) and that satisfies

$$F(x_0, x_1, \ldots, x_n) = \phi\left(x_0,\ \sqrt{{x_1}^2+\cdots+{x_n}^2}\right). \tag{2.19}$$

for $(x_0,\ldots,x_n)$ sufficiently near the origin in $\mathbb{R}^{n+1}$. (The reason for defining ϕ as an even function of $\sigma = \sqrt{{x_1}^2+\cdots+{x_n}^2}$ rather than directly as a function of ${x_1}^2+\cdots+{x_n}^2$ is that it leads to a more manageable equation in the uniqueness analysis to be done below.)

Because ϕ is even in its second argument, the quotient $\phi_\sigma(t,\sigma)/\sigma$ is a real-analytic function (also even in σ) on $V \subset \mathbb{R}$. Thus, the graph of F can be parametrized analytically near $0 \in \mathbb{C}^{n+1}$ in the form

$$\left(t + \mathrm{i}\,\phi_t(t,|x|),\ x_1 + \mathrm{i}\,x_1\frac{\phi_\sigma(t,|x|)}{|x|},\ \ldots,\ x_n + \mathrm{i}\,x_n\frac{\phi_\sigma(t,|x|)}{|x|}\right) \tag{2.20}$$

where $|x| = \sqrt{{x_1}^2+\cdots+{x_n}^2}$. The condition that $T_0L = \mathbb{R}^{n+1}$ is then equivalent to the conditions that $\phi_t(t,\sigma)$ have vanishing differential at $(t,\sigma) = (0,0)$ and that the smooth function $\phi_\sigma(t,\sigma)/\sigma$ vanish at $(t,\sigma) = (0,0)$.

An immediate consequence of the representation (2.20) is that, in a neighborhood of the origin, $L \cap \mathbb{C}$ consists of the points of the form

$$(\,t + \mathrm{i}\,\phi_t(t,0),\ 0, \ldots, 0) \tag{2.21}$$

for $|t|$ sufficiently small, which is a nonsingular real-analytic curve.

Since z was an arbitrary point of $L \cap \mathbb{C}$, it follows that $L \cap \mathbb{C}$ is a nonsingular embedded real-analytic curve $A \subset \mathbb{C}$.

Finally, suppose that $A = L \cap \mathbb{C}$ has a compact component, i.e., an embedded closed curve $A_0 \subset A$. Choose a periodic, nonsingular parametrization $(x,y) : \mathbb{R} \to A_0$ with period 1, i.e., $x(t{+}1) = x(t)$ and $y(t{+}1) = y(t)$. There will then exist a smooth function $\theta : \mathbb{R} \to \mathbb{R}$ such that

$$x'(t) = \cos\theta(t)\,\sqrt{x'(t)^2 + y'(t)^2}, \qquad y'(t) = -\sin\theta(t)\,\sqrt{x'(t)^2 + y'(t)^2}. \tag{2.22}$$

Since A_0 is an embedded curve, it has rotation number ± 1, i.e., $\theta(t{+}1) = \theta(t) \pm 2\pi$. By reversing the orientation of the parametrization if necessary, it can be supposed that $\theta(t{+}1) = \theta(t) + 2\pi$.

Now, the special Lagrangian planes that contain the tangent line to A at $z = \bigl(x(0), y(0)\bigr)$ are, by construction, of the form $P_{\theta(0)/n+k\pi/n}$ for $k = 0, 1, \ldots, n{-}1$, so one of these is T_zL. Fix k so that $T_zL = P_{\theta(0)/n+k\pi/n}$. Then, by continuity, it follows that

$$T_{\left(x(t),y(t)\right)}L = P_{\theta(t)/n+k\pi/n} \tag{2.23}$$

for all t. However, since x and y are periodic of period 1, it would then follow that

$$\begin{aligned} P_{\theta(0)/n+k\pi/n} &= T_{\left(x(0),y(0)\right)}L = T_{\left(x(1),y(1)\right)}L \\ &= P_{\theta(1)/n+k\pi/n} = P_{\theta(0)/n+2\pi/n+k\pi/n}. \end{aligned} \tag{2.24}$$

However, if n were greater than 2, the first and last of the planes in this string of equalities could not be equal. Thus, $n = 2$, as claimed. □

Example 3 (Extending the unit circle). The last statement of Proposition 1 may seem surprising at first. However, consider the case in which A is the unit circle in $\mathbb{C} \subset \mathbb{C}^{n+1}$. Using the circle invariance to reduce the integration problem to an ODE (as in Example 1), one finds that the locus

$$L = \left\{ (z, \zeta\mathbf{u}) \,\middle|\, |\zeta|^2 = n\big(|z|^2 - 1\big),\ \mathrm{Re}(z\zeta^n) = 0,\ \mathbf{u} \in S^{n-1} \right\} \subset \mathbb{C}^{n+1} \tag{2.25}$$

is smooth away from $A = L \cap \mathbb{C}$ and is special Lagrangian. Near each point of A, the locus L is a union of n smooth distinct sheets intersecting pairwise in A.

When n is odd, $L \setminus A$ is connected, so L is analytically irreducible. When n is even, $L \setminus A$ has two connected components, and, in fact, L is the union of two distinct analytically irreducible pieces: L_+, on which $\mathrm{Im}(z\zeta^n)$ is nonnegative, and L_-, on which $\mathrm{Im}(z\zeta^n)$ is nonpositive. Near each point of A, each of L_+ and L_- is a union of $\frac{1}{2}n$ smooth distinct sheets. Only when $n = 2$ are L_+ and L_- smooth.

This behavior for the unit circle is typical for embedded closed curves in general, as will be seen.

Remark 3 (Invariance under λ^n). The representation (2.20) also shows that F is even in the variables $x_1, \ldots, x_n$. In particular, this implies that, if $L \subset \mathbb{C}^{n+1}$ is the graph of F regarded as a special Lagrangian submanifold, then $\lambda^n \star L = L$ on some neighborhood of $0 \in \mathbb{C}^{n+1}$.

By analytic continuation, it follows that, for any embedded, non-singular, SO(n)-invariant, connected special Lagrangian $L \subset \mathbb{C}^{n+1}$ that meets $\mathbb{C}$, the submanifolds $\lambda^n \star L$ and L are equal in some open neighborhood of $L \cap \mathbb{C}$.

Remark 4 (Image in M_0). It is a consequence of the proof that there is a neighborhood $U \subset \mathbb{C}^{n+1}$ of 0 such that $U \cap L = \Phi(\Sigma_\phi \times S^{n-1})$ where $\Sigma_\phi \subset \mathbb{C}^2$ is the analytically embedded surface

$$\Sigma_\phi = \left\{ \big(t + \mathrm{i}\,\phi_t(t,\sigma),\ \sigma + \mathrm{i}\,\phi_\sigma(t,\sigma) \big) \,\middle|\, (t,\sigma) \in V \right\}. \tag{2.26}$$

Note that, when n is odd, the embedding $\iota_\phi : V \to \mathbb{C}^2$ defined by

$$\iota_\phi(t,\sigma) = \big(t + \mathrm{i}\,\phi_t(t,\sigma),\ \sigma + \mathrm{i}\,\phi_\sigma(t,\sigma) \big) \tag{2.27}$$

pulls back ω_1 (which is only defined on M_0 via the formulae (2.10)) to a complex-valued 1-form that extends smoothly across the curve $\sigma = 0$. This is not so surprising since, when n is odd, the mapping $(w, \zeta) \mapsto (w, -\zeta)$ is J_0-antilinear on M_0 and ι_ϕ intertwines this mapping with the orientation reversing mapping $(t,\sigma) \mapsto (t, -\sigma)$ on V. In any case, ${\iota_\phi}^*(\omega_1)$ is a $(1,0)$-form for a natural complex structure on V that makes ι_ϕ into a J_0-complex curve away from the locus $\sigma = 0$.

The picture when n is even is slightly more complicated and is left to the reader.

3. Local uniqueness

Suppose now that $A \subset \mathbb{C}$ is a connected, nonsingular real-analytic curve. The goal now is to determine whether A is the fixed locus of a nonsingular SO(n)-invariant special Lagrangian (n+1)-fold $L \subset \mathbb{C}^{n+1}$ and, if so, in how many ways.

Remark 5 (Lack of uniqueness). As has already been remarked, if $A = L \cap \mathbb{C}$ for some embedded nonsingular SO(n)-invariant special Lagrangian (n+1)-fold L, then $A = \lambda^j \star L \cap \mathbb{C}$ for any integer j in the range $0 \le j < n$. Moreover, since L is embedded, A has an open neighborhood $U \subset \mathbb{C}^{n+1}$ so that

$$\lambda^j \star L \cap \lambda^k \star L \cap U = A \tag{3.1}$$

for any j and k satisfying $0 \le j < k < n$.

One consequence of the analysis to be done below is that any embedded nonsingular special Lagrangian $(n{+}1)$-fold $L' \subset \mathbb{C}^{n+1}$ that contains A agrees with $\lambda^j{\star}L$ in some open neighborhood of A in $\mathbb{C}^{n+1}$ for some integer j in the range $0 \le j < n$.[7] This result will follow from general considerations once it is shown that a local version of this uniqueness holds.

3.1. **Reduction to an equation.** Using the action (2.17), to understand the local picture, it suffices to understand the case where the curve A passes through the origin $0 \in \mathbb{C}^{n+1}$, its tangent there is spanned by $\partial/\partial x_0$, and $T_0L = \mathbb{R}^{n+1}$ is spanned by $\partial/\partial x_0$, $\partial/\partial x_1$, …, $\partial/\partial x_n$.

As in §2.4, it follows that, in a neighborhood of $0 \in \mathbb{C}^{n+1}$, L can be described as a graph of the form (2.18) for some function F of the form (2.19), where ϕ is a real-analytic function on a neighborhood of $0 \in \mathbb{R}^2$.

By hypothesis, A can be parametrized near $0 \in \mathbb{C}^{n+1}$ in the form

$$\big(t + \mathrm{i}\, f_0'(t), 0, \dots, 0\big) \tag{3.2}$$

for some function f_0 that is real-analytic in a neighborhood of $0 \in \mathbb{R}$ and abides $f_0'(0) = f_0''(0) = 0$. The function f_0 can be made unique by requiring that $f_0(0) = 0$. (The reason for starting with f_0' instead of f_0 should be apparent.)

Now, the condition that L contain A becomes the condition

$$\phi(t, 0) = f_0(t). \tag{3.3}$$

(Bear in mind the normalizations $F(0, \dots, 0) = f_0(0) = 0$.) Furthermore, the condition that $T_0L = \mathbb{R}^{n+1}$ implies

$$\phi_{\sigma\sigma}(0, 0) = 0. \tag{3.4}$$

Let σ stand for $\sqrt{{x_1}^2 + \cdots + {x_n}^2}$ and let t stand for x_0 as the coordinates in the domain of ϕ in $\mathbb{R}^2$. The condition that the 1-graph of F as defined in (2.18) be special Lagrangian is, of course, a second order partial differential equation on F. This equation can be expressed in terms of ϕ in the form

$$\mathrm{Im}\,\big((\sigma + \mathrm{i}\,\phi_\sigma)^{n-1}\, \mathrm{d}(\sigma + \mathrm{i}\,\phi_\sigma) \wedge \mathrm{d}(t + \mathrm{i}\,\phi_t)\big) = 0, \tag{3.5}$$

or, in more classical PDE terms:

$$\mathrm{Im}\,\big((\sigma + \mathrm{i}\,\phi_\sigma)^{n-1}\big((1 + \mathrm{i}\,\phi_{tt})(1 + \mathrm{i}\,\phi_{\sigma\sigma}) + {\phi_{\sigma t}}^2\big)\big) = 0, \tag{3.6}$$

which is a singular, second order Monge-Ampère equation that is elliptic when $\sigma \ne 0$ but degenerate along $\sigma = 0$.

Proposition 2. *Let f_0 be a real-analytic function defined on an open interval containing $0 \in \mathbb{R}$ and that satisfies $f_0(0) = f_0'(0) = f_0''(0) = 0$. Then there is at most one real-analytic function ϕ defined on a neighborhood of $(0, 0) \in \mathbb{R}^2$ that satisfies $\phi(t, \sigma) = \phi(t, -\sigma)$, the equation (3.6), and the initial conditions (3.3) and (3.4).*

Proof. Any such ϕ must have a power series expansion of the form

$$\phi(t, \sigma) = f_0(t) + \tfrac{1}{2!} f_1(t)\sigma^2 + \tfrac{1}{4!} f_2(t)\sigma^4 + \tfrac{1}{6!} f_3(t)\sigma^6 + \cdots . \tag{3.7}$$

[7] Recall from Remark 3 that $\lambda^{j+n}{\star}L$ and $\lambda^j{\star}L$ agree in some open neighborhood of A. Thus, one can restrict the range of j to $0 \le j < n$ as claimed.

where the f_i for $i > 0$ are real-analytic on some interval $|t| \leq \tau$ and satisfy a bound of the form $|f_i(t)| \leq C/M^i$ when $|t| < \tau$ for some constants $C > 0$ and $M > 0$. Moreover, by (3.4), f_1 must satisfy $f_1(0) = 0$.

To prove uniqueness, it suffices to show that the equation (3.6) together with the specified initial conditions determine f_i uniquely for $i > 0$. Write (3.6) in the equivalent form

$$\operatorname{Im}\left(\left(1+\mathrm{i}\,\frac{\phi_\sigma}{\sigma}\right)^{n-1}\left((1+\mathrm{i}\,\phi_{tt})(1+\mathrm{i}\,\phi_{\sigma\sigma})+{\phi_{\sigma t}}^2\right)\right)=0, \tag{3.8}$$

Now, substituting (3.7) into (3.8), collecting like powers of σ, and considering the coefficient of σ^0 yields

$$\operatorname{Im}\left(\left(1+\mathrm{i}\,f_1(t)\right)^n\left(1+\mathrm{i}\,f_0''(t)\right)\right)=0. \tag{3.9}$$

This is an algebraic equation for f_1 in terms of f_0'' that has up to n distinct roots. However, by hypothesis and initial condition, $f_0''(0) = f_1(0) = 0$, so there is only one continuous choice of f_1 that will satisfy the given initial condition, namely

$$f_1(t) = -\tan\left(\frac{\tan^{-1}\left(f_0''(t)\right)}{n}\right). \tag{3.10}$$

Thus, assume henceforth that f_1 is defined by (3.10). For simplicity, set

$$R(t) = \left(1+\mathrm{i}\,f_1(t)\right)^n\left(1+\mathrm{i}\,f_0''(t)\right) \tag{3.11}$$

and note that R is real-valued and satisfies $R(0) = 1$.

Let $\tau > 0$ be such that f_0, f_1, and $1/R$ have convergent power series in the interval $-\tau < t < \tau$.

Now, the derivatives of ϕ that appear on the left hand side of (3.8) have convergent power series expansions in σ of the form

$$\begin{aligned} \phi_{tt}(t,\sigma) &= \sum_{k=0}^{\infty}\frac{1}{(2k)!}f_k''(t)\sigma^{2k}, & \phi_{\sigma t}(t,\sigma) &= \sum_{k=0}^{\infty}\frac{1}{(2k+1)!}f_{k+1}'(t)\sigma^{2k+1}, \\ \phi_{\sigma\sigma}(t,\sigma) &= \sum_{k=0}^{\infty}\frac{1}{(2k)!}f_{k+1}(t)\sigma^{2k}, & \frac{\phi_\sigma(t,\sigma)}{\sigma} &= \sum_{k=0}^{\infty}\frac{1}{(2k+1)!}f_{k+1}(t)\sigma^{2k}. \end{aligned} \tag{3.12}$$

Using these expressions, the formula for f_1 and R, and the definitions given above, it follows that the left hand side of (3.8) has a series expansion in σ of the form

$$\sum_{k=1}^{\infty}\left[\frac{R}{1+{f_1}^2}\,\frac{2k+n}{(2k+1)!}\,f_{k+1} - Q_k(f_1,\dots,f_k,f_1',\dots,f_k',f_0'',\dots,f_k'')\right]\sigma^{2k} \tag{3.13}$$

where each Q_k is an explicit polynomial of total degree at most $n{+}1$ in its $3k{+}1$ arguments as listed. (The essential points to note are: First, the coefficient of σ^0 has already been set to zero by the definition of f_1. Second, as the terms on the right hand sides of (3.12) show, for $k > 0$, the coefficient of σ^{2k} in the σ-expansion of the left hand side of (3.8) is a sum of products of terms whose coefficients only involve the quantities $f_1, \dots, f_k, f_{k+1}, f_1', \dots, f_k', f_0'', \dots, f_k''$. Moreover, since terms involving f_{k+1} in this sum can only come from terms that have a factor of $f_{k+1}\sigma^{2k}$, the remaining factors in such a term must occur as coefficients of σ^0. These are easily collected and computed, leading to the expression given in (3.13).)

Consequently, since f_0 is given and f_1 is determined by (3.10), the functions f_{k+1} for $k \geq 1$ are determined recursively by the equations

$$f_{k+1} = \frac{1+{f_1}^2}{R}\,\frac{(2k+1)!}{2k+n}\,Q_k(f_1,\ldots,f_k,f_1',\ldots,f_k',f_0'',\ldots,f_k''). \tag{3.14}$$

In particular, each f_k has a convergent power series on the interval $-\tau < t < \tau$.

Thus, there is a unique formal power series solution $\phi(t,\sigma)$ to (3.6) that is even in σ and satisfies the initial conditions (3.3) and (3.4), as was to be shown. □

4. Local existence

While the uniqueness theorem above demonstrates the existence of a formal power series solution to (3.6) that satisfies the appropriate initial conditions, proving that the formal series converges on some neighborhood of $(t,\sigma) = (0,0)$ in $\mathbb{R}^2$ is not easy to do directly. The crude argument used to derive the recursion formula (3.14) does not give sufficient detail about the polynomials Q_k to allow any effective estimates to be done on the growth of the terms f_k as k tends to ∞.

Moreover, because of the singular nature of the equations involved, a direct appeal to the Cauchy-Kowalewski theorem does not seem to be feasible.

4.1. **An existence theorem of Gérard and Tahara.** However, by making use of a more subtle application of the method of majorants, Gérard and Tahara have proven an existence and uniqueness theorem in the holomorphic category that suffices to prove that the above series (which is the only formal power series solution) does, in fact, converge. Their main existence result can be found as Theorem 8.0.3 of their book [2] and concerns the existence of holomorphic solutions of holomorphic singular partial differential equations. For the convenience of the reader, this result will now be summarized in the case of second order equations, which is all that will be needed for this article.

Their existence theorem applies to certain singular partial differential equations for a function $u = u(t,\sigma)$ of the form

$$G\big(\,t,\,\sigma,\,u,\,u_t,\,\sigma u_\sigma,\,u_{tt},\,\sigma u_{\sigma t},\,\sigma^2 u_{\sigma\sigma}\,\big) = 0, \tag{4.1}$$

where G is a real-analytic function of its eight arguments in a neighborhood of the origin in $\mathbb{R}^8$.

To avoid confusion in the statement of their result, it will be important to adopt a notation that clearly distinguishes the arguments and suggests their meanings. For this reason, the eight arguments of G will be written as

$$G\big(t,\sigma,Z_{0,0},Z_{0,1},Z_{1,0},Z_{0,2},Z_{1,1},Z_{2,0}\big). \tag{4.2}$$

The idea is that $Z_{k,l}$ is the name of the variable into which the derivative expression

$$\sigma^k\,\frac{\partial^{k+l}u}{\partial^k\sigma\,\partial^l t} \tag{4.3}$$

is to be substituted in (4.2) in order to form the left hand side of (4.1) Also, in order to save writing, for any analytic function on a neighborhood of the origin in $\mathbb{R}^8$, say

$$H\big(t,\sigma,Z_{0,0},Z_{0,1},Z_{1,0},Z_{0,2},Z_{1,1},Z_{2,0}\big), \tag{4.4}$$

the expression $H(t,\mathbf{0})$ will be an abbreviation for $H(t,0,0,0,0,0,0,0)$.

Theorem 1 (cf. Theorem 8.0.3 of [2])**.** *Let* $G = G\big(t,\sigma,Z_{0,0},Z_{0,1},Z_{1,0},Z_{0,2},Z_{1,1},Z_{2,0}\big)$ *be a real-analytic function defined on a neighborhood of the origin in* $\mathbb{R}^8$*. Suppose that* G *satisfies the following conditions:*

(1) $G(t,\mathbf{0}) \equiv 0$,

(2) $\dfrac{\partial G}{\partial Z_{0,1}}(t,\mathbf{0}) \equiv \dfrac{\partial G}{\partial Z_{1,1}}(t,\mathbf{0}) \equiv \dfrac{\partial G}{\partial Z_{0,2}}(t,\mathbf{0}) \equiv 0$,

(3) $\dfrac{\partial G}{\partial Z_{2,0}}(0,\mathbf{0}) \neq 0$,

(4) $\dfrac{\partial G}{\partial Z_{2,0}}(0,\mathbf{0})\,k^2 + \dfrac{\partial G}{\partial Z_{1,0}}(0,\mathbf{0})\,k + \dfrac{\partial G}{\partial Z_{0,0}}(0,\mathbf{0}) \neq 0$ *for any integer* $k>0$.

Then there is a unique real-analytic function u *defined on an open neighborhood of* $(t,\sigma) = (0,0) \in \mathbb{R}^2$ *that satisfies the equation*

$$G\big(\,t,\,\sigma,\,u,\,u_t,\,\sigma u_\sigma,\,u_{tt},\,\sigma u_{\sigma t},\,\sigma^2 u_{\sigma\sigma}\,\big)= 0 \tag{4.5}$$

and the initial condition $u(t,0) \equiv 0$.

Remark 6 (The hypotheses of Gérard and Tahara). The reader may well wonder about the significance of the hypotheses listed above.

The first two conditions are meant to make the initial condition $u(t,0) = 0$ formally compatible with the equation (4.1).

The third condition ensures that the equation (4.1) can be solved near the origin in $\mathbb{R}^8$ for the expression $\sigma^2 u_{\sigma\sigma}$. In fact, Gérard and Tahara state their theorem for equations of the form

$$\sigma^2 u_{\sigma\sigma} - F\big(\,t,\,\sigma,\,u,\,u_t,\,\sigma u_\sigma,\,u_{tt},\,\sigma u_{\sigma t}\,\big)= 0 \tag{4.6}$$

where F is a real-analytic function of its arguments on a neighborhood of the origin in $\mathbb{R}^7$. The form in which it has been stated here is more convenient for the intended application.

The fourth condition ensures that, when one tries to solve (4.1) by substituting in a power series of the form

$$u(t,\sigma) = f_1(t)\,\sigma + f_2(t)\sigma^2 + \cdots\,, \tag{4.7}$$

and setting equal to zero the coefficients of the various powers of σ that result, the resulting conditions on the f_i can be resolved into a recursion relation for the functions f_i without having to divide by any quantity that can vanish. In other words, the fourth condition guarantees that there will exist a formal power series solution of the form (4.7).

Remark 7 (Extensions). Theorem 8.0.3 of [2] is considerably more general than the result stated here as Theorem 1. In the first place, Gérard and Tahara consider equations of arbitrary (finite) order, not just second order equations. In the second place, their existence and uniqueness theorem provides for the classification of more general solutions than just the single-valued real-analytic solutions. However, the specialized version of their theorem stated above is all that will be needed in this article.

4.2. **Application.** The results of Gérard and Tahara can now be applied to show that the series solution ϕ found in the previous section converges.

Proposition 3. *Let f_0 be a real-analytic function defined on an open interval containing $0 \in \mathbb{R}$ and that satisfies $f_0(0) = f_0'(0) = f_0''(0) = 0$. Then there exists a real-analytic function ϕ defined on a neighborhood of $(0,0) \in \mathbb{R}^2$ that satisfies $\phi(t,\sigma) = \phi(t,-\sigma)$, the equation* (3.6)*, and the initial conditions* (3.3) *and* (3.4)*.*

Proof. It has already been shown that if the equation (3.6) has a real-analytic solution ϕ that is even in σ and satisfies the initial conditions (3.3) and (3.4), then its expansion in powers of σ, must be given by (3.7) where f_1 is defined by (3.10) and f_k for $k \geq 2$ are then determined recursively via (3.14). Existence will follow once this series is shown to converge.

This suggests looking for a solution to (3.6) of the form

$$\phi(t,\sigma) = f_0(t) + \frac{1}{2}\big(f_1(t) + u(t,\sigma)\big)\sigma^2 \tag{4.8}$$

where $f_1(t)$ is defined in terms of the real-analytic function f_0 via (3.10) and where u is a real-analytic function defined on a neighborhood of the origin in $\mathbb{R}^2$ that satisfies $u(t,0) \equiv 0$.

Now, substituting (4.8) into the equation (3.8) yields an equation for u of the form (4.1) where G is taken to be the analytic function

$$\begin{aligned} G = \operatorname{Im}\Big\{ &(1 + \mathrm{i}(f_1(t){+}2Z_{0,0}{+}Z_{1,0}))^{n-1}\Big[\sigma^2(f_1'(t){+}2Z_{0,1}{+}Z_{1,1})^2 \\ &+ \big(1 + \mathrm{i}(f_0''(t){+}\tfrac{1}{2}(f_1''(t){+}Z_{0,2})\sigma^2)\big)\big(1{+}\mathrm{i}(f_1(t){+}2Z_{0,0}{+}4Z_{1,0}{+}Z_{2,0})\big)\Big]\Big\}. \end{aligned} \tag{4.9}$$

In order to apply Theorem 1, the four hypotheses on G must now be verified.

To verify the first hypothesis, note that

$$G(t,\mathbf{0}) = \operatorname{Im}\big(\,(1 + \mathrm{i}f_1(t))^n\,(\,1 + \mathrm{i}f_0''(t)\,)\big). \tag{4.10}$$

Of course, f_1 was defined so that the expression on the right hand side would vanish identically.

To verify the second hypothesis, note that each of the variables $Z_{0,1}$, $Z_{1,1}$ and $Z_{0,2}$ occur in the formula (4.9) with a coefficient that is a positive power of σ. Of course, this immediately implies that

$$\frac{\partial G}{\partial Z_{0,1}}(t,\mathbf{0}) \equiv \frac{\partial G}{\partial Z_{1,1}}(t,\mathbf{0}) \equiv \frac{\partial G}{\partial Z_{0,2}}(t,\mathbf{0}) \equiv 0. \tag{4.11}$$

The third and fourth hypotheses follow from the easily derived formulae

$$\frac{\partial G}{\partial Z_{2,0}}(0,\mathbf{0}) = 1\,, \quad \frac{\partial G}{\partial Z_{1,0}}(0,\mathbf{0}) = n{+}3\,, \quad \frac{\partial G}{\partial Z_{0,0}}(0,\mathbf{0}) = 2n\,. \tag{4.12}$$

Thus, Theorem 1 applies: There exists a unique real-analytic function u defined on a neighborhood of the origin in $\mathbb{R}^2$ that satisfies the equation (4.1) where G is defined as in (4.9) and the initial condition $u(t,0) \equiv 0$.

Because G as defined in (4.9) is an even function of σ, the function v defined by $v(t,\sigma) = u(t,-\sigma)$ also satisfies (4.1). Since $v(t,0) \equiv 0$, the uniqueness part of Theorem 1 implies that $v(t,\sigma) = u(t,\sigma)$, i.e., that $u(t,-\sigma) = u(t,\sigma)$.

Finally, using this solution u to define ϕ via (4.8), the desired local existence is established. In particular, the series for ϕ defined via (3.10) and (3.14) must be convergent on some neighborhood of $(0,0) \in \mathbb{R}^2$. $\square$

5. Conclusions

In this last section, the local PDE results of the previous sections will be applied to prove the main results.

The first result is an 'n-uniqueness' theorem:

Theorem 2. *Suppose that L and L' are nonsingular, embedded* $\mathrm{SO}(n)$*-invariant special Lagrangian submanifolds of* $\mathbb{C}^{n+1}$ *that have the same fixed locus* $A\subset\mathbb{C}$ *and that, moreover,* A *is connected.*

Then there is an open A-neighborhood $U\subset\mathbb{C}^{n+1}$ and a unique integer j satisfying $0\le j<n$ such that

$$L'\cap U=\lambda^j{\star}L\cap U. \tag{5.1}$$

Moreover, when $0\le j<k<n$, the submanifolds $\lambda^j{\star}L\cap U$ and $\lambda^k{\star}L\cap U$ intersect only along A.

Proof. If A is empty, then there is nothing to prove, so assume that A is nonempty and that $z=(z_0,0,\dots,0)\in A$. Both T_zL and T_zL' are special Lagrangian $(n{+}1)$-planes that contain the line T_zA and hence there is an integer j in the range $0\le j<n$ and an angle ψ in the range $0\le\psi<\pi$ such that $T_zL=P_\psi$ and $T_zL'=P_{\psi+(j/n)\pi}$.

By applying a motion in the group generated by the mappings (2.17), it can be assumed that $z=0\in\mathbb{C}^{n+1}$ and that $T_zL=\mathbb{R}^{n+1}=P_0$. Then $T_z(\lambda^{-j}{\star}L')=T_zL=P_0$ as well.

Since L and L' are embedded, it follows that there is an open neighborhood W_z of $0\in\mathbb{R}^{n+1}$ and an open neighborhood U_z of $0\in\mathbb{C}^{n+1}$ such that there exist unique real-analytic functions F and F' defined on W_z and vanishing at $0\in\mathbb{R}^{n+1}$ so that

$$L\cap U_z=\left\{\left(x_0+\mathrm{i}\,\tfrac{\partial F}{\partial x_0},\,\dots,x_n+\mathrm{i}\,\tfrac{\partial F}{\partial x_n}\right)\,\middle|\,(x_0,\dots,x_n)\in W_z\right\} \tag{5.2}$$

while

$$\lambda^{-j}{\star}L'\cap U_z=\left\{\left(x_0+\mathrm{i}\,\tfrac{\partial F'}{\partial x_0},\,\dots,x_n+\mathrm{i}\,\tfrac{\partial F'}{\partial x_n}\right)\,\middle|\,(x_0,\dots,x_n)\in W_z\right\}. \tag{5.3}$$

Since $T_z(\lambda^{-j}{\star}L')=T_zL=\mathbb{R}^{n+1}$, it follows that F and F' have vanishing first and second derivatives at $0\in\mathbb{R}^{n+1}$. As was argued in the proof of Proposition 1, it follows that there unique exist real-analytic functions ϕ and ϕ' defined on a neighborhood of $(0,0)$ in $\mathbb{R}^2$ such that $\phi(0,0)=\phi'(0,0)=0$ and such that

$$\begin{aligned}F(x_0,x_1,\dots,x_n)&=\phi\big(x_0,\sqrt{x_1{}^2{+}\cdots{+}x_n{}^2}\big)\\ F'(x_0,x_1,\dots,x_n)&=\phi'\big(x_0,\sqrt{x_1{}^2{+}\cdots{+}x_n{}^2}\big)\end{aligned} \tag{5.4}$$

when $x_0{}^2{+}\cdots{+}x_n{}^2$ is sufficiently small. Since $L\cap U_z\cap\mathbb{C}=L'\cap U_z\cap\mathbb{C}=A\cap U_z$, it follows that $\phi(t,0)=\phi'(t,0)$ for $|t|$ sufficiently small. Moreover, by construction, ϕ and ϕ' are solutions of (3.6) that are even in the second variable (i.e., σ). Because $T_z(\lambda^{-j}{\star}L')=T_zL=\mathbb{R}^{n+1}$, it follows that $\phi_{\sigma\sigma}(0,0)=\phi'_{\sigma\sigma}(0,0)=0$ and, setting $f_0(t)=\phi(t,0)=\phi'(t,0)$, that $f_0(0)=f_0'(0)=f_0''(0)=0$.

Now Proposition 2 implies that $\phi=\phi'$, which, in turn, implies that $L\cap U_z=\lambda^{-j}{\star}L'\cap U_z$.

Since z was chosen arbitrarily in A, it has been established that every $z\in A$ has an open neighborhood $U_z\subset\mathbb{C}^{n+1}$ such that there exists a unique integer j_z satisfying $0\le j_z<n$ so that $L'\cap U_z=(\lambda^{j_z}{\star}L)\cap U_z$. Moreover, by shrinking U_z appropriately, it can be arranged that, when $0\le j<k \mod n$, the two manifolds $(\lambda^j{\star}L)$ and $(\lambda^k{\star}L)\cap U_z$ only intersect along $A\cap U_z$.

In particular, it follows that the integer j_z is locally constant in z and hence, since A is connected, it follows that there is a unique integer j in the range $0 \leq j < n$ such that $j_z = j$ for all $z \in A$.

Now let U be the union of the U_z as z ranges over A. Then U and j have the required properties. □

The second main result is an existence result:

Theorem 3. *Suppose that $A \subset \mathbb{C}$ is an embedded, connected real-analytic curve and that either A is noncompact or else that $n = 2$. Then there exists an embedded, connected special Lagrangian submanifold $L \subset \mathbb{C}^{n+1}$ that is $\mathrm{SO}(n)$-invariant and satisfies $L \cap \mathbb{C} = A$.*

Proof. Consider the set B that consists of the pairs (z, P_ψ) with $z \in A$ such that P_ψ is a special Lagrangian plane of the form (2.15) that contains the line T_zA. By the discussion in §2.3, it follows that B is a $\mathbb{Z}_n$-bundle over A.

If A is noncompact, then this bundle is trivial and there exists a continuous (in fact, real-analytic) $\psi : A \to \mathbb{R}$ such that $(z, P_{\psi(z)})$ is a section of B over all of A. If A is compact and $n = 2$, then the argument given at the end of the proof of Proposition 1 shows that, because the rotation number of A is ± 1, the bundle B is trivial in this case as well, so that, again, there is a mapping $\psi : A \to \mathbb{R}/(\pi\mathbb{Z})$ so that that $(z, P_{\psi(z)})$ is a section of B over all of A.

Now, Proposition 3, together with the assumption that A is embedded, implies that every point $z \in A$ has an open neighborhood $U_z \subset \mathbb{C}^{n+1}$ that intersects A in a connected arc $U_z \cap A$ and in which there exists an embedded, nonsingular, $SO(n)$-invariant special Lagrangian submanifold, say $L_z \subset U_z$ such that $L_z \cap \mathbb{C} = A \cap U_z$. By Theorem 2, this L_z can be made unique by requiring that $T_z(L_z) = P_{\psi(z)}$.

By the connectedness of $U_z \cap A$, and the continuity of ψ, it follows that $T_w(L_z) = P_{\psi(w)} = T_w(L_w)$ for any $w \in U_z \cap A$. Consequently, $T_y(L_z) = T_y(L_w)$ for all $y \in U_z \cap U_w \cap A$. It then follows from Theorem 2 that L_z and L_w are equal in some open neighborhood of the interval $U_z \cap U_w \cap A$.

It is now not difficult to conclude that there exists an open neighborhood U of A itself and an embedded, nonsingular, $SO(n)$-invariant special Lagrangian submanifold $L \subset \mathbb{C}^{n+1}$ that agrees with L_z on some open neighborhood of z for each $z \in A$. □

References

[1] I. Castro and F. Urbano, *On a new construction of special Lagrangian immersions in complex Euclidean space*, preprint 2004. (To appear in the Quarterly Journal of Mathematics.) math.DG/0403108 6

[2] R. Gérard and H. Tahara, *Singular nonlinear partial differential equations*, Aspects of Mathematics, Friedr. Vieweg & Sohn, Braunschweig, 1996. MR 2001c:35056 14, 15

[3] F. R. Harvey and H. B. Lawson, Jr., *Calibrated geometries*, Acta Math. **148** (1982), 47–157. MR 1985i:53058 2

[4] D.D. Joyce, *Lectures on special Lagrangian geometry*, preprint, 2001. math.DG/0111111 2, 5

[5] ———, U(1)*-invariant special Lagrangian* 3*-folds. I. Nonsingular solutions*, preprint 2001. (To appear in Advances in Mathematics.) math.DG/0111324 5

[6] ———, U(1)-*invariant special Lagrangian 3-folds. II. Existence of singular solutions*, preprint, 2001. math.DG/0111326 5

[7] ———, U(1)-*invariant special Lagrangian 3-folds. III. Properties of singular solutions*, preprint, 2002. math.DG/0204343 5

[8] C. B. Morrey, Jr., *On the analyticity of the solutions of analytic non-linear elliptic systems of partial differential equations. I. Analyticity in the interior*, Amer. J. Math., **80**, (1958), 198–218. MR 21:5070 9

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